A purity theorem for configuration spaces of smooth compact algebraic varieties

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Abstract

B. Totaro showed [10] that the rational cohomology of configuration spaces of smooth complex projective varieties is isomorphic as an algebra to the E_2 term of the Leray spectral sequence corresponding to the open embedding of the configuration space into the Cartesian power. In this note we show that the isomorphism can be chosen to be compatible with the mixed Hodge structures. In particular, we prove that the mixed Hodge structures on the configuration spaces of smooth complex projective varieties are direct sums of pure Hodge structures.

1 Introduction

For a topological space X we define the *ordered configuration spaces* $F(X, n)$ by

$$F(X, n) = \{(x_1, \ldots, x_n) \in X^{\times n} \mid x_i \neq x_j \text{ if } i \neq j\}.$$

J. Morgan constructed in [6] a rational homotopy model for the complement U of a normal crossing divisor $D_1 \cup \ldots \cup D_m$ in a smooth compact algebraic variety V; Morgan's model is determined by $H^*(V, \mathbb{Q})$, the rational cohomology of normalisations of all possible intersections $D_{i_1} \cap \ldots D_{i_l}$ and the Gysin maps. Using the results of [6], W. Fulton and R. MacPherson [3] construted a rational homotopy model of $F(X, n)$ when X is a smooth compact algebraic variety. This conctruction was simplified by I. Kriz [4]. B. Totaro independently obtained the same commutative dg-algebra as I. Kriz by a completely different method, namely as the E_2 term of the Leray spectral sequence of the embedding $F(X, n) \subset X^{\times n}$.

One can use both J. Morgan's and B. Totaro's methods to compute the mixed Hodge numbers of $F(X, n)$ for a smooth compact X. However, unfortunately, neither approach allows one to compute the rational mixed Hodge structures, as shown by the following example:

Example. Let C be a smooth compact curve of genus > 0 and let X be the complement of a finite set Y. The E_2 term of the Leray spectral sequence of $j : X \subset C$ will be isomorphic to $H^*(X, \mathbb{Q})$ as an algebra, but not as a mixed Hodge structure, since the Hodge structure on

E_2 is a direct sum of pure ones and the Hodge structure on X will often be mixed (and will depend on more than the Gysin map $H^*(Y, \mathbb{Q}) \to H^{*+2}(C, \mathbb{Q})$).

The purpose of this note is to show that this does not happen if we take the open embedding of a configuration space into the corresponding Cartesian power.

Theorem 1. *Let X be smooth compact complex algebraic variety. Then $H^*(F(X,n), \mathbb{Q})$ decomposes as a direct sum of pure Hodge structures.*

Corollary 1. *$H^*(F(X,n), \mathbb{Q})$ is isomorphic to the E_3 term of the Leray spectral sequence of the open embedding $F(X,n) \subset X^{\times n}$ both as an algebra and a mixed Hodge structure.*

The same remains true if we remove not the whole of the fat diagonal, but just some components.

2 Preliminaries

In this section we recall some general constructions related to mixed Hodge complexes of sheaves and of vector spaces. In particular, we describe an explicit mixed Hodge complex (of vector spaces) which computes $H^*(F(X,n), \mathbb{Q})$ as a mixed Hodge structure for a smooth complete X.

For a manifold Y we denote the algebra of smooth $\mathbb{C}$-valued forms on Y by $\mathcal{E}^*(Y)$.

Recall that one of the ways to introduce a Hodge structure on the cohomology of a smooth complete variety X is as follows: the data $\mathcal{H}dg^*(X)$ consisting of the constant sheaf $\underline{\mathbb{Q}}_X$, the de Rham complex of sheaves $\underline{\Omega}^*_X$ equipped with the "dumb" filtration and the natural morphism $\underline{\mathbb{Q}}_X \to \underline{\Omega}^*_X$ is a Hodge complex of sheaves (see e.g., [8, 2.3.3]). Applying the $R\Gamma$ functor to $\mathcal{H}dg^*(X)$, one obtains a Hodge complex of vector spaces whose cohomology carries a natural pure Hodge structure. The algebra $\mathcal{E}^*(X)$ is equipped with a Hodge filtration induced by the decomposition of forms into (p,q)-types. Due to the $\partial\bar{\partial}$-lemma (see next section), the Hodge filtration is strictly compatible with the differential.

Proposition 1. *Let $\mathcal{E}^*_{PL,C^\infty}(X)$ and $A^*_{PL}(Y)$ be the algebra of all piecewise C^∞ forms (respectively the algebra of Sullivan $\mathbb{Q}$-polynomial forms on Y [9]), on X with respect to some triangulation of X. Then the data consisting of $A^*_{PL}(X)$, $\mathcal{E}^*(Y)$ and the morphism in the bounded derived category of $\mathbb{Q}$-vector spaces induced by $A^*_{PL}(X) \subset \mathcal{E}^*_{PL,C^\infty}(X) \supset \mathcal{E}^*(Y)$ represents $R\Gamma\mathcal{H}dg^*(X)$ in the category of Hodge complexes of vector spaces.*

$\square$

If an algebraic variety D is the union of smooth compact varieties $D_1, \ldots, D_m$ such that all (set theoretical) intersections $D_J = \bigcap_{j \in J} D_j$, $J \subset \{1, \ldots, m\}$ are smooth, then one can glue a mixed Hodge complex of sheaves (see e.g. [8, 3.3]) $\mathcal{H}dg^*(D)$ out of $\mathcal{H}dg^*(D_J)$'s as follows. If

$J_1 \subset J_2$ are subsets of $\{1, \dots, m\}$, there is a morphism of sheaves $i_{J_1,J_2} : i_*^{J_1}\underline{\mathbb{Q}}_{D_{J_1}} \to i_*^{J_2}\underline{\mathbb{Q}}_{D_{J_2}}$ where i^{J_1} and i^{J_2} are the inclusions of D_{J_1} and D_{J_2} into D.

Set K_* to be the complex of sheaves given by $K_i = \bigoplus_{J,\#J=i+1} i_*^J\underline{\mathbb{Q}}_{D_J}$; the differential on $i_*^J\underline{\mathbb{Q}}_{D_J} \subset K_i$ is

$$\sum_{j\notin J}(-1)^{\#\{k\in J|k<j\}-1}i_{J,J\cup\{j\}}.$$

Set ${}_KW^iK_* = \bigoplus_{j\geq i} K_j$ i.e. ${}_KW^*$ is the dumb filtration on K_*; define ${}_KW_iK_* = {}_KW^{-i}K_*$. Let L_* be the complex of sheaves defined in the same way as K_* using not the constant sheaves $\underline{\mathbb{Q}}_{D_J}$ but the de Rham complexes of sheaves $\underline{\Omega}^*_{D_J}$. The complex L_* admits two natural filtrations: the weight filtration ${}_LW_*$ is defined exactly as for K_* and the Hodge filtration F^* is the direct sum of the dumb filtrations on each $\underline{\Omega}^*_{D_J}$. Note that there is a natural "comparison" morphism of complexes of sheaves $c : K_* \to L_*$ which is compatible with the weight filtrations and which becomes a quasi-isomorphism after tensoring the source by $\mathbb{C}$.

Proposition 2. *The data $(K_*, (K, {}_KW_*), \mathrm{id}_K, (L, {}_LW_*, F^*), c)$ is a mixed Hodge complex of sheaves. The corresponding mixed Hodge complex of vector spaces gives the mixed Hodge structure on $H^*(D, \mathbb{Q})$.*

$\square$

Suppose now that a variety D of the above form is embedded into a smooth complete variety Y. Then we can apply the mixed cone construction (see e.g. [8, theorem 3.22]) to obtain the mixed Hodge structure on $H^*(Y, D, \mathbb{Q})$. We describe only the resulting mixed Hodge complex of vector spaces. Set $A^{*,*}(Y, D, C^\infty)$ to be the double complex defined as follows.

$$A^{0,*}(Y, D, C^\infty) = \mathcal{E}^*(Y),$$
$$A^{i,*}(Y, D, C^\infty) = \bigoplus_{0\leq j_1<\cdots<j_i\leq m} \mathcal{E}^*(D_{j_1} \cap \cdots \cap D_{j_i}), i > 0.$$

The differential on $A^{*,*}(Y, D, C^\infty)$ is defined as $d' + \delta$, where d' acts on the i-th column as $(-1)^i \cdot$ the de Rham differential, $\delta\omega = \bigoplus \omega|_{D_i} \in A^{1,*}(Y, D, C^\infty)$ for $\omega \in A^{0,*}(Y, D, C^\infty)$, and

$$(\delta\omega)_{j_1,\dots,j_{i+1}} = \sum(-1)^{l+1}\omega_{j_1,\dots,\hat{j}_l,\dots,j_{i+1}}$$

for

$$\omega = \bigoplus_{0\leq j_1<\cdots<j_i\leq \frac{n(n-1)}{2}} \omega_{j_1,\dots,j_i} \in A^{i,*}(Y, D, C^\infty), i > 0.$$

Define the weight filtration by ${}_{C^\infty}W_iA^{*,*}(Y, D, C^\infty) = \sum_{j\geq -i} A^{j,*}(Y, D, C^\infty)$ and set the F^p term of the Hodge filtration to be the direct sum of the F^p terms on all $A^{*,*}(Y, D, C^\infty)$'s.

Triangulate Y so that each D_i is a subpolyhedron. Then there are double complexes $A^{*,*}(Y, D, PL)$ and $A^{*,*}(Y, D, PL, C^\infty)$ constructed exactly as $A^{i,*}(Y, D, C^\infty)$, but using piecewise $\mathbb{Q}$-polynomial forms and piecewise smooth complex forms respectively. The weight filtration ${}_{PL}W_*$ on $A^{*,*}(Y, D, PL)$ is introduced in the same way as on $A^{i,*}(Y, D, C^\infty)$.

For a double complex $(C^{*,*})$ we denote the corresponding total complex $s[C^{*,*}]$ by C^*, so $A^*(Y, D, PL) = s[A^{*,*}(Y, D, PL)]$ and $A^*(Y, D, C^\infty) = s[A^{*,*}(Y, D, C^\infty)]$. There are obvious comparison morphisms of double complexes, hence of their total complexes

$$A^*(Y, D, PL) \subset A^*(Y, D, PL, C^\infty) \supset A^*(Y, D, C^\infty)$$

which induce a morphism $c : A^*(Y, D, PL) \to A^*(Y, D, C^\infty)$ is the bounded W-filtered derived category of $\mathbb{Q}$-vector spaces. (See e.g. [8, A.3.1] for more details on filtered derived categories; briefly, in these categories one inverts not all filtration preserving quasi-isomorphisms but only filtered quasi-isomorphisms.)

Proposition 3. *The data*

$$(A^*(Y, D, PL), (A^*(Y, D, PL), {}_{PL}W_*), \mathrm{id}_{A^*(Y,D,PL)}, (A^*(Y, D, C^\infty), {}_{C^\infty}W, F^*), c)$$

is a mixed Hodge complex of vector spaces which calculates $H^*(Y, D, \mathbb{Q})$.

□

3 ∂- and $\bar\partial$-closed forms

The results of the previous section apply to the complement of a family of smooth subvarieties in a complete variety such that the intersection of any number of subvarieties in the family is smooth. In the case when the ambient variety is the Cartesian power $X^{\times n}$ of a smooth complete variety X and the subvarieties are $D_{ij} = \{(x_1, \dots, x_n) \in X^{\times n} \mid x_i = x_j\}$ there is another mixed Hodge complex, which we describe in this section. Later we show that the complex we are about to construct is quasi-isomorphic to yet another one whose cohomology is manifestly a direct sum of pure Hodge structures.

For a complex analytic variety Y let $\mathcal{E}^*_\partial(Y)$ and $\mathcal{E}^*_{\bar\partial}(Y)$ be the subalgebras of ∂-closed and $\bar\partial$-closed forms respectively. We say that the $\partial\bar\partial$-lemma holds for Y, if

$$\ker\bar\partial \cap \mathrm{Im}\,\partial = \mathrm{Im}\,\partial\bar\partial = \ker\partial \cap \mathrm{Im}\,\bar\partial.$$

Lemma 1. *Suppose the $\partial\bar\partial$-lemma holds for Y. Then*

1. *The differential $d = \bar\partial$ on $\mathcal{E}^*_\partial(Y)$ is strictly compatible with the Hodge filtration. (The Hodge filtration on $\mathcal{E}^*_\partial(Y)$ induced from the Hodge filtration on $\mathcal{E}^*(Y)$.)*

2. *The inclusion $\mathcal{E}^*_\partial(Y) \to \mathcal{E}^*(Y)$ is an algebra quasi-isomorphism. The resulting cohomology map induces an isomorphism of the Hodge filtrations.*

3. *There is an algebra quasi-isomorphism $\mathcal{E}^*_\partial(Y) \to H^*(Y, \mathbb{C})$.*

This is essentially standard, see [1]. Nevertheless we give a proof, as for our purposes it is more convenient to work with ∂-closed forms and not $\bar\partial$-closed forms.

Proof. Part 1 is straightforward. In fact, this is the reason we use ∂- rather than $\bar\partial$-closed forms.

To prove part 2 it would suffice to show that the E_1-terms of the spectral sequences associated to the Hodge filtrations on $\mathcal{E}^*_\partial(Y)$ and $\mathcal{E}^*(Y)$ are isomorphic. To do that it would suffice, in turn, to prove that for every given p the inclusion of complexes

$$(\mathcal{E}^{p,*}_\partial(Y), \bar\partial) \subset (\mathcal{E}^{p,*}(Y), \bar\partial) \tag{1}$$

is a quasi-isomorphism. Here

$$\mathcal{E}^{p,*}_\partial(Y) = (\mathcal{E}^{p,q}_\partial(Y)) = (\mathcal{E}^{p,q}(Y) \cap \mathcal{E}^*_\partial(Y)).$$

Let us show that (1) induces a surjective map in cohomology. Take an $\omega \in \mathcal{E}^{p,q}(Y)$ such that $\bar\partial\omega = 0$. Using the $\partial\bar\partial$-lemma we find an $\omega_1 \in \mathcal{E}^{p,q-1}(Y)$ such that $\partial\omega = \partial\bar\partial\omega_1$. So $\partial(\omega - \bar\partial\omega_1) = 0$, and so ω is $\bar\partial$- homologous to a class $\in \mathcal{E}^{p,*}_\partial(Y)$.

Similarly, let us show that (1) induces an injective map in cohomology. Take an $\omega \in \mathcal{E}^{p,q}_\partial(Y)$. We have $\partial\omega = 0$. Suppose there is an $\omega_1 \in \mathcal{E}^{p,q-1}(Y)$ such that $\omega = \bar\partial\omega_1$. We use the $\partial\bar\partial$-lemma again to find an $\omega_2 \in \mathcal{E}^{p-1,q-1}$ such that $\omega = \bar\partial\partial\omega_2$. Since $\partial\omega_2 \in \mathcal{E}^{p,q-1}_\partial(Y)$, we see that ω is homologous to 0 in $\mathcal{E}^{p,*}_\partial(Y)$.

Let us now prove part 3. We first construct a projection map $p_Y : \mathcal{E}^*_\partial(Y) \to H^*(Y, \mathbb{C})$ follows: take an $\omega \in \mathcal{E}^{p,q}_\partial(Y)$, find (using the $\partial\bar\partial$-lemma) an $\eta \in \mathcal{E}^{p-1,q}(Y)$ such that $\bar\partial(\omega + \partial\eta) = d(\omega + \partial\eta) = 0$ and take ω to the cohomology class of $\omega + \partial\eta$. Let us show that $p_Y(\omega)$ does not depend on η. Suppose $\eta' \in \mathcal{E}^{p-1,q}(Y)$ is another form such that $\bar\partial(\omega + \partial\eta) = 0$. We use the $\partial\bar\partial$-lemma again to find an $\omega' \in \mathcal{E}^{p-1,q-1}(Y)$ such that $\partial(\eta - \eta') = \partial\bar\partial\omega'$. Then $d(\frac{1}{2}(\bar\partial\omega' - \partial\omega')) = \partial(\eta - \eta')$.

We now prove that p_Y is a chain map. To do that we need to show that if an $\omega \in \mathcal{E}^{p,q}_\partial(Y)$ is $\bar\partial\omega_1$ for an $\omega_1 \in \mathcal{E}^{p,q-1}_\partial(Y)$, then ω is a coboundary in $\mathcal{E}^*(Y)$. Using the $\partial\bar\partial$-lemma we find an $\omega' \in \mathcal{E}^{p-1,q-1}(Y)$ such that $\omega = \bar\partial\partial\omega'$ and then proceed as in the previous paragraph.

Next we show that p_Y is an algebra homomorphism: for $\omega_1 \in \mathcal{E}^{p_1,q_1}_\partial(Y), \omega_2 \in \mathcal{E}^{p_2,q_2}_\partial(Y), \eta_1 \in \mathcal{E}^{p_1-1,q_1}(Y), \eta_2 \in \mathcal{E}^{p_2-1,q_2}(Y)$ we have

$$(\omega_1 + \partial\eta_1) \wedge (\omega_2 + \partial\eta_2) = \omega_1 \wedge \omega_2 + \partial(\eta_1 \wedge \omega_2 \pm \omega_1 \wedge \eta_2 + \eta_1 \wedge \partial\eta_2).$$

Moreover, if $\omega_1 + \partial\eta_1$ and $\omega_2 + \partial\eta_2$ are $\bar\partial$-closed, so is their product.

Finally, we need to show that p_Y induces an isomorphism of cohomology rings. This follows from part 2 and the fact that if an $\omega \in \mathcal{E}^{p,q}_{\partial}(Y)$ is a $\bar\partial$-cocycle, then $p_Y(\omega)$ is the class of ω in $H^*(\mathcal{E}^*(Y)) = H^*(Y, \mathbb{C})$. $\square$

The $\partial\bar\partial$-lemma holds for any compact Kähler manifold [1], hence, for any smooth complex projective variety. If the $\partial\bar\partial$-lemma holds for Y' and $Y' \to Y$ is a holomorphic birational mapping with Y smooth and compact, then $\partial\bar\partial$-lemma holds also for Y [7]. Let now Y be an arbitrary smooth compact complex algebraic variety. There exists a smooth projective variety Y' and an surjective morphism $Y' \to Y$ which is a birational isomorphism [5], so the $\partial\bar\partial$-lemma holds for Y, and so the map p_Y is well defined and both the inclusion map $i_Y : \mathcal{E}^*_{\partial}(Y) \subset \mathcal{E}^*(Y)$ and $p_X : \mathcal{E}^*_{\partial}(Y) \to H^*(Y, \mathbb{C})$ are algebra quasi-isomorphisms.

We now want to construct a mixed Hodge complex of vector spaces similar to the one from the previous section which would calculate the groups $H^*(X^{\times n}, D, \mathbb{Q})$, where

$$D = \bigcup_{1 \le i < j \le n} D_{ij},$$

using ∂-closed, rather than all smooth forms. We would also like to replace the algebra $A^*_{PL}(X)$ of piecewise polynomial forms by its minimal model, which we denote by $\mathcal{M}_X$. Let us fix a comparison quasi-isomorphism $\mu : \mathcal{M}_X \to A^*_{PL}(X)$.

We will write cdga for "commutative differential graded algebra". Recall the definition of a homotopy between $\mathbf{k}$-cdga morphisms (here $\mathbf{k}$ is a field). The algebra $\Omega^*(\mathbf{k})$ of polynomial differential forms on $\mathbf{k}^1$ has two natural augmentations $\varepsilon_i : \Omega^*(\mathbf{k}) \to \mathbf{k}, i = 1, 2$: the restrictions to 0 and 1. We say that two $\mathbf{k}$-cdga morphisms $f_1, f_2 : A \to B$ are *homotopic* if there if a $\mathbf{k}$-cdga map $F : A \to B \otimes \Omega^*(\mathbf{k})$ such that $f_i, i = 1, 2$ is equal to F composed with $\mathrm{id}_B \otimes \varepsilon_i : B \otimes \Omega^*(\mathbf{k}) \to B \otimes \mathbf{k} = B$.

Here is one of the advantages of replacing $A^*_{PL}(X)$ by $\mathcal{M}_X$: we have a cdga map $f : \mathcal{M}_X \to \mathcal{E}^*_{PL,C^\infty}(X)$ which is the composition of μ and the inclusion $A^*_{PL}(X) \subset \mathcal{E}^*_{PL,C^\infty}(X)$. Using the lifting lemma [2, lemma 12.4] and the "surjective trick" (ibid., p. 148) we obtain the following

Proposition 4. *There is a cdga map $\nu : \mathcal{M}_X \to \mathcal{E}^*_{\partial}(X)$ such that f is homotopic to the composition of ν and the inclusion $\mathcal{E}^*_{\partial}(X) \subset \mathcal{E}^*_{PL,C^\infty}(X)$.*

$\square$

By taking tensor powers of ν and composing with the maps

$$(\mathcal{E}^*_{\partial}(X))^{\otimes k} \to \mathcal{E}^*_{\partial}(X^{\times k}), \tag{2}$$

and similarly for μ we get for every k a diagram which commutes up to cgda homotopy:

$$\begin{array}{ccc} A_{PL}(X^k) & \longrightarrow & \mathcal{E}_{PL,C^\infty}(X^k) \\ \uparrow & & \uparrow \\ \mathcal{M}_X^{\otimes k} & \longrightarrow & \mathcal{E}^*_{\partial}(X^{\times k}) \end{array}$$

We are now ready to introduce another mixed Hodge complex of vector spaces. For every Y and D as in the previous section we construct a double complex $A^{*,*}(Y,D,C^\infty,\partial)$ in the same way as the complexes $A^{*,*}(Y,D,C^\infty)$, $A^{*,*}(Y,D,PL,C^\infty)$ and $A^{*,*}(Y,D,PL)$ on p. 2 using ∂-closed differential forms.

Now let us take $Y = X^{\times n}$ and $D =$ the fat diagonal, i.e. $D = \bigcup D_{i,j}$ where $1 \le i < j \le n$ and

$$D_{i,j} = \{(x_1,\dots,x_n) \in X^{\times n} \mid x_i = x_j\}.$$

The procedure described on p. 2 then produces a double complex starting from an arbitrary cdga B as follows: Note that every intersection of D_i's is isomorphic to a Cartesian power $X^{\times k}$ and if $I \subset \{1,\dots,\frac{n-1}{2}\}$ and $J = I$ minus one element, then the map induced by the inclusion $\bigcap_{j\in J} D_j \subset \bigcap_{i\in I} D_i$ is either the identity or the identity times the diagonal embedding $X \subset X^{\times 2}$. So if one replaces forms on every $\bigcap_{i\in I} D_i \cong X^k$ by $B^{\otimes k}$ and the map of forms induced by $\bigcap_{j\in J} D_j \subset \bigcap_{i\in I} D_i$ by the identity tensor the multiplication map $B \times B \to B$, one obtains a double complex which we will denote $A^{*,*}(B)$.

Note that the following diagram commutes on the nose (i.e., not just up to homotopy):

$$\begin{tikzcd}
\mathcal{M}_{\mathcal{X}} \arrow[rr, "\nu"] & & \mathcal{E}^*_\partial(X) \\
{\mathcal{M}_{\mathcal{X}}}^{\otimes 2} \arrow[u] \arrow[r] & (\mathcal{E}^*_\partial(X))^{\otimes 2} \arrow[r] & \mathcal{E}^*_\partial(X^{\times 2}) \arrow[u]
\end{tikzcd}$$

Here the arrow on the left is multiplication, the arrow on the right is induced by the diagonal embedding and the lower horizontal arrows are $\nu^{\otimes 2}$ and (2). So we get comparison chain maps

$$A^{*,*}(\mathcal{M}_X) \to A^{*,*}(\mathcal{E}^*_\partial(X)) \to A^{*,*}(Y,D,C^\infty,\partial).$$

The composition of these will be denoted c'. It becomes a quasi-isomorphism if we tensor $A^{*,*}(\mathcal{M}_X)$ by $\mathbb{C}$.

Also, we equip $A^{*,*}(\mathcal{M}_X)$, $A^{*,*}(\mathcal{E}^*_\partial(X))$ and $A^{*,*}(Y,D,C^\infty,\partial)$ with weight filtrations in the same way as this was done for $A(Y,D,C^\infty)$ etc. in the previous section. These will be denoted ${}_{\mathcal{M}}W$ and ${}_{C^\infty,\partial}W$, or simply W when no confusion is likely. Note that we can also introduce a similar weight filtration on $A^{*,*}(B)$ for any cdga B. This filtration will be denoted ${}_BW$ or simply W.

Note also that $A^{*,*}(\mathcal{E}^*_\partial(X))$ and $A^{*,*}(Y,D,C^\infty,\partial)$ have Hodge filtrations; these will be denoted by F.

Same as in section 2, we denote the total complex of a double complex $C^{*,*}$ by C^*.

Proposition 5. *Both sets of data*

$$(A^*(M),(A^*(M),{}_{\mathcal{M}}W),\mathrm{id}_{A^*(M)},(A^*(Y,D,C^\infty,\partial),{}_{C^\infty,\partial}W,F),c')$$

and

$$(A^*(M), (A^*(M), {}_{\mathcal{M}}W), \mathrm{id}_{A^*(M)}, (A^*(\mathcal{E}_\partial(X)), {}_{C^\infty,\partial}W, F), c'')$$

are mixed Hodge complexes of $\mathbb{Q}$-vector spaces. Here the second comparison map c'' is the functor $A^*(-)$ from Lemma 2 applied to ν.

□

4 Proof of Theorem 1

We now need to compare the mixed Hodge complexes from Propositions 3 and 5.

Lemma 2. *The correspondence $B \mapsto (A^{*,*}(B), {}_BW)$ is a functor from the category of cdga's to the category of filtered double complexes. This functor takes homotopic cgda maps to filtered homotopic maps of complexes.*

□

In order to compare the mixed Hodge complexes we use the following diagram:

$$\begin{tikzcd}
A^*(\mathcal{M}_X) \arrow[rr, "\mu"] \arrow[d] & & A^*(Y, D, C^\infty, \partial) \arrow[d] \\
A^*(Y, D, PL) \arrow[r] & A^*(Y, D, C^\infty, PL) & A(Y, D, C^\infty) \arrow[l]
\end{tikzcd} \tag{3}$$

This diagram commutes up to filtered homotopy (with respect to the weight filtrations). To see this consider this diagram:

$$\begin{tikzcd}
 & A^*(\mathcal{M}_X) \arrow[dl] \arrow[dr] & \\
A^*(\mathcal{E}^*_\partial(X)) \arrow[r] \arrow[d] & A^*(\mathcal{E}^*_{C^\infty,PL}(X)) \arrow[d] & A^*(A_{PL}(X)) \arrow[l] \arrow[d] \\
A^*(Y, D, C^\infty, \partial) \arrow[r] & A^*(Y, D, C^\infty, PL) & A^*(Y, D, PL) \arrow[l]
\end{tikzcd}$$

The bottom squares here commute on the nose and the upper square commutes up to filtered homotopy with respect to the weight filtrations by Lemma 2. So we conclude that (3) commutes up to filtered homotopy.

The cohomology of a mixed Hodge complex carries a mixed Hodge structure. The arrows on the right and on the left of (3) induce cohomology isomorphisms compatible with the weight and Hodge filtrations. So they induce isomorphisms of mixed Hodge structures (this statement

is certainly false if we consider filtered vector spaces but it is true for mixed Hodge structures, see [8, 3.1]).

We now construct a third mixed Hodge complex of vector spaces. Namely, we take the mixed Hodge complex from Proposition 5 and replace ∂-closed forms with cohomology. More specifically, we compose the map $p_X : \mathcal{E}^*_\partial(X) \to H^*(X,\mathbb{C})$ from Lemma 1 with $\nu : \mathcal{M}_X \to \mathcal{E}^*_\partial(X)$. The image of the resulting map will be $\subset H^*(X,\mathbb{Q})$, so we get a cdga map $\nu_\mathbb{Q} : \mathcal{M}_X \to H^*(X,\mathbb{Q})$. We also have a comparison map

$$\bar{c} : A(H^*(X,\mathbb{Q})) \to A(H^*(X,\mathbb{C}))$$

which is obtained by applying the functor from Lemma 2 to $H^*(X,\mathbb{Q}) \to H^*(X,\mathbb{C})$.

The data

$$(A^*(H^*(X,\mathbb{Q})), (A^*(H^*(X,\mathbb{Q})), W), \mathrm{id}_{A^*(H^*(X,\mathbb{Q}))}, (A^*(H^*(X,\mathbb{C})), W, F), \bar{c}) \tag{4}$$

is a mixed Hodge complex of vector spaces. The maps $A^*(\nu_\mathbb{Q})$ and $A^*(p_X)$ induce an isomorphism of mixed Hodge structures between the cohomology of second mixed Hodge complex from Lemma 5 and the cohomology of (4). The latter is a direct sum of pure ones. $\square$

References

[1] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, *Real homotopy type of Kähler manifolds*, Invent. Math. 29 (1975), 245–274.

[2] Y. Felix, S. Halperin, J.-C. Thomas, *Rational homotopy theory*, Springer, Graduate Texts in Mathematics 205, 2001.

[3] W. Fulton, R. MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2), 139 (1994), no. 1, 183–225.

[4] I. Kriz, *On the rational homotopy type of configuration spaces*, Ann. of Math. (2), 139 (1994), no. 2, 227–237.

[5] B. G. Moishezon, *On n-dimensional compact complex manifolds having n algebraically independent meromorphic functions*, (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 30 (1966), part I 133–174, part II 345–386, part III 621–656.

[6] J. Morgan, *The algebraic topology of smooth algebraic varieties*, Inst. Hautes Études Sci. Publ. Math., no. 48 (1978), 137–204.

[7] A. N. Parshin, *On a certain generalization of Jacobian manifold*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 30 1966 175–182.

[8] C. A. M. Peters, J. H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik.

[9] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. No. 47 (1977), 269–331 (1978).

[10] B. Totaro, *Configuration spaces of algebraic varieties*, Topology 35 (1996), no. 4, 1057–1067.

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